Journal of Computational and Applied Mathematics 234 (2010) 2062–2068

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Determination of a material constant in the impedance boundary condition for electromagnetic fields

Viera Zemanová [a,*], Marián Slodička [a], Luc Dupré [b]

[a] Department of Mathematical Analysis, Ghent University, Galglaan 2, 9000 Ghent, Belgium

[b] Department Electrical Energy, Systems and Automation, Ghent University, Sint-Pietersnieuwstraat 41, 9000 Ghent, Belgium

ARTICLE INFO

Article history:
Received 17 September 2008
Received in revised form 9 February 2009

Keywords:
Inverse problem
Electromagnetism
Impedance boundary condition

ABSTRACT

We study a recovery problem for an unknown boundary data at the boundary part Γ_{loss} in static electromagnetism. Our computational area is a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz continuous boundary. The problem for determining the coefficient λ is considered. This coefficient represents one of the ferromagnetic material characteristics occupying this domain. The existence and uniqueness of a weak solution are proved and a numerical method for its recovery is supported by numerical experiments.

1. Introduction and physical motivation

For the design of electromagnetic devices, an accurate evaluation of the material characteristics, like the electromagnetic loss P, the permeability μ and the electrical conductivity σ, of their magnetic circuit is essential. The importance originate from the increasing requirements set for high performance devices. Classically, the electromagnetic loss of magnetic materials are quantified by means of standard measurement equipment, enforcing a time dependent magnetic field to the body of the test sample. For this type of measurement equipment, one obtains the iron losses P in the ferromagnetic material under investigation starting from two sensor signals. The first signal is related to the time dependent magnetic field $\boldsymbol{H}$ enforced at the surface of the material body while the second signal defines the time dependent magnetic flux in the material. The latter is directly related to the induced electrical field $\boldsymbol{E}$ at the surface of the material body. The loss originates from the eddy currents present in the material.

The measured losses are also related to the electromagnetic fields at the surface of the body of the test sample through the pointing vector $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$. Indeed, considering the surface A of the body of the material, one has for the iron loss:

$$P = \oint_A (\boldsymbol{E} \times \boldsymbol{H}) \nu \, ds. \tag{1}$$

The phasors of the electrical field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$ at the boundary of the material are given by

$$\nabla \times \boldsymbol{H} \times \nu = \lambda \boldsymbol{H} \times \nu \times \nu. \tag{2}$$

To derive the precise mathematical model we start with Maxwell's equations

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -\partial_t \boldsymbol{B}, \\ \nabla \times \boldsymbol{H} &= \sigma \boldsymbol{E} + \epsilon \partial_t \boldsymbol{E}, \\ \nabla \cdot (\epsilon \boldsymbol{E}) &= \rho, \\ \nabla \cdot \boldsymbol{B} &= 0, \end{aligned} \tag{3}$$

* Corresponding author.
E-mail addresses: viera.zemanova@ugent.be (V. Zemanová), marian.slodicka@ugent.be (M. Slodička), luc.dupre@ugent.be (L. Dupré).

0377-0427/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2009.08.064

where ϵ is a permittivity of medium. There exists as well the relation between the magnetic field $\boldsymbol{H}$ and the magnetic flux density $\boldsymbol{B}$. In case of linear material we assume $\boldsymbol{B} = \mu \boldsymbol{H}$. In many applications we assume quasi-static equations, it means, one field is static, i.e., $\partial_t \boldsymbol{E} \approx \mathbf{0}$, the other one changes dynamically.

After the elimination of $\boldsymbol{E}$ in (3), we obtain the following model for the magnetic field

$$\sigma\mu\partial_t \boldsymbol{H} + \nabla \times \nabla \times \boldsymbol{H} = \boldsymbol{0}. \tag{4}$$

Next we consider an equidistant partitioning with a time step $\tau = \frac{T}{n}$, for any $n \in \mathbb{N}$. Thus, we divide the time interval $[0, T]$ into n sub-intervals $[t_{i-1}, t_i]$ for $t_i = i\tau$. Let us apply the time discretization based on backward Euler's method and we obtain the following recurrent system:

$$\sigma\mu\tau^{-1}\boldsymbol{H}^n + \nabla \times \nabla \times \boldsymbol{H}^n = \sigma\mu\tau^{-1}\boldsymbol{H}^{n-1}. \tag{5}$$

A direct problem of this type is usually accompanied by one of the following standard boundary condition

$$\boldsymbol{H}^n \times \boldsymbol{\nu} = \boldsymbol{a} \qquad \text{or} \qquad \nabla \times \boldsymbol{H}^n \times \boldsymbol{\nu} = \boldsymbol{b},$$

which are prescribed almost everywhere at the boundary. By employing relation (2) the physical importance of another type of the boundary condition arises.

2. Problem formulation

A ferromagnetic occupies a bounded domain $\Omega \subset \mathbb{R}^3$ with a Lipschitz continuous boundary Γ split into three complementary, non-empty and non-overlapping parts, $\Gamma = \overline{\Gamma}_{\text{Dir}} + \overline{\Gamma}_{\text{Neu}} + \overline{\Gamma}_{\text{loss}}$. The outward normal to Γ is denoted by $\boldsymbol{\nu}$. The object of interest is to identify a coefficient λ describing the electromagnetic losses. More exactly, the following inverse steady-state eddy-current problem is studied:

Problem 1. Find $(\lambda, \boldsymbol{H}_\lambda) \in (\mathbb{R}_+, \mathbf{H}(\text{curl}\,; \Omega))$ such that

$$\begin{aligned}
K\boldsymbol{H}_\lambda + \nabla \times \nabla \times \boldsymbol{H}_\lambda &= \boldsymbol{f} && \text{in } \Omega \\
\boldsymbol{H}_\lambda \times \boldsymbol{\nu} &= \boldsymbol{0} && \text{on } \Gamma_{\text{Dir}} \\
\nabla \times \boldsymbol{H}_\lambda \times \boldsymbol{\nu} &= \boldsymbol{g} && \text{on } \Gamma_{\text{Neu}} \\
\nabla \times \boldsymbol{H}_\lambda \times \boldsymbol{\nu} &= \lambda(\boldsymbol{H}_\lambda \times \boldsymbol{\nu} \times \boldsymbol{\nu}) && \text{on } \Gamma_{\text{loss}}
\end{aligned}$$

with given $K, \boldsymbol{f}$ and $\boldsymbol{g}$. To see easier the connection with parameter λ our solution will be from now on denoted by $\boldsymbol{H}_\lambda$. [The equation in the domain in Problem 1 has the same form as relation (5).]

Due to physical reasons we assume that the right-hand side of the eddy-current equation in Problem 1 is divergence free, i.e.

$$\nabla \cdot \boldsymbol{f} = 0. \tag{6}$$

2.1. Methodology

At the first glance, Problem 1 appears to be similar to the following one:

Problem 2. Find $(h, u) \in (\mathbb{R}_+, H^1(\Omega))$ such that

$$\begin{aligned}
pu + \nabla \cdot (-K\nabla u) &= f && \text{in } \Omega \\
u &= 0 && \text{on } \Gamma_{\text{Dir}} \\
-K\nabla u \cdot \boldsymbol{\nu} &= g && \text{on } \Gamma_{\text{Neu}} \\
-K\nabla u \cdot \boldsymbol{\nu} &= hu && \text{on } \Gamma_{\text{loss}}
\end{aligned}$$

where K, p, f and g are given data.

The equations in both problems are the same. This can be easily seen using the well-known identity

$$-\triangle \boldsymbol{H} = \nabla \times \nabla \times \boldsymbol{H} - \nabla(\nabla \cdot \boldsymbol{H})$$

and taking into account the fact that $\nabla \cdot \boldsymbol{B} = 0$, so then $\nabla \cdot \boldsymbol{H} = 0$ as well. The difference between both the cases, Problems 1 and 2 lies in the regularity of corresponding solutions.

The recovery of the Robin coefficient, convective transfer coefficient, from the overspecified data in linear steady-state elliptic boundary value problem 2, was done in [1]. The identification was based on the difference between the outside and the inside temperature on Γ_{non}, in sense of $\mathbf{L}_2(\Gamma_{\text{non}})$. Slodička and Van Keer consider the whole boundary accessible, but on the part Γ_{non}, i.e. nonaccessible part of the boundary, the data are not 'precise', they are known in an 'average' sense only.

A very similar problem setting can be obtained from the problem of corrosion detection (cf. [2]). Here, the author works in a thin plate while thick domains cause the instabilities of the numerical approach. The data of the problem consist of prescribed current flux and voltage measurements on an accessible part of the specimen boundary. The inverse problem is to determine the quantitative information about corrosion occurring on an inaccessible part of domain.

3. Assumptions

Problem 1 can have infinite number of solutions according to a free positive parameter λ at $\Gamma_{\rm loss}$. Our goal is to design such an additional boundary condition, called iron loss boundary condition, which will guarantee the uniqueness of a solution. As to be shown later this can be ensured by the following side condition

$$0 < M = \int_{\Gamma_{\rm loss}} |\boldsymbol{H}_\lambda \times \boldsymbol{\nu}|^2 < \lim_{\lambda\to 0_+} m(\lambda). \tag{7}$$

The function $m(\lambda)$ will be specified later.

The following conditions on data are assumed:

$$\begin{aligned} &0 < K_{\min} \le K \le K_{\max}, \quad \text{a.e. in } \Omega, \\ &\boldsymbol{f} \in \mathbf{L}_2(\Omega), \qquad \boldsymbol{g} \in \mathbf{L}_2(\Gamma_{\rm Neu}). \end{aligned} \tag{8}$$

We shall work in a variational framework. By $(\boldsymbol{w}, \boldsymbol{z})$, resp. $(\boldsymbol{w}, \boldsymbol{z})_\Gamma$, the usual $\mathbf{L}_2$-inner product of any real- or vector-valued functions $\boldsymbol{w}$ and $\boldsymbol{z}$ in Ω, resp. on the boundary Γ, is denoted, i.e. $(\boldsymbol{w}, \boldsymbol{z}) = \int_\Omega \boldsymbol{w} \cdot \boldsymbol{z}$ and $\|\boldsymbol{w}\| = \sqrt{(\boldsymbol{w}, \boldsymbol{w})}$, resp. $(\boldsymbol{w}, \boldsymbol{z})_\Gamma = \int_\Gamma \boldsymbol{w} \cdot \boldsymbol{z}$. The standard function spaces $\mathbf{H}(\mathrm{curl}\,; \Omega)$ and $\mathbf{L}_p(\Gamma)$ for some $p > 1$, see [3], are to be used. The norm in $\mathbf{H}(\mathrm{curl}\,; \Omega)$ is defined as

$$\|\boldsymbol{\varphi}\|^2_{\mathbf{H}(\mathrm{curl}\,;\Omega)} = \|\boldsymbol{\varphi}\|^2 + \|\nabla \times \boldsymbol{\varphi}\|^2 .$$

The space of test functions is denoted by

$$\boldsymbol{V} = \{\boldsymbol{\varphi} \in \mathbf{H}(\mathrm{curl}\,; \Omega);\ \boldsymbol{\varphi} \times \boldsymbol{\nu} = \boldsymbol{0} \text{ on } \Gamma_{\rm Dir}\} .$$

This is a natural choice for our Problem 1. $\boldsymbol{V}$ is a reflexive Banach space endowed with the standard norm $\|\cdot\|_{\mathbf{H}(\mathrm{curl}\,;\Omega)}$.

For ease of exposition, $\boldsymbol{g} = \boldsymbol{0}$ is set. Then the variational formulation of Problem 1 reads as

$$K(\boldsymbol{H}_\lambda, \boldsymbol{\varphi})_\Omega + (\nabla \times \boldsymbol{H}_\lambda, \nabla \times \boldsymbol{\varphi})_\Omega + \lambda(\boldsymbol{H}_\lambda \times \boldsymbol{\nu}, \boldsymbol{\varphi} \times \boldsymbol{\nu})_{\Gamma_{\rm loss}} = (\boldsymbol{f}, \boldsymbol{\varphi})_\Omega \tag{9}$$

for any $\boldsymbol{\varphi} \in \boldsymbol{V}$.

Now the question is, where is the information about the electromagnetic loss P hidden within the latter formulation. Intuitively the boundary term will be analysed and consequently the equality

$$(\nabla \times \boldsymbol{H}_\lambda, \varphi \times \boldsymbol{\nu})_{\Gamma_{\rm loss}} = \lambda(\boldsymbol{H}_\lambda \times \boldsymbol{\nu}, \varphi \times \boldsymbol{\nu})_{\Gamma_{\rm loss}}$$

will be achieved. Setting $\varphi = \boldsymbol{H}_\lambda$ and recalling that $\nabla \times \boldsymbol{H} = \boldsymbol{E}$ yields to

$$(\boldsymbol{E}, \boldsymbol{H}_\lambda \times \boldsymbol{\nu})_{\Gamma_{\rm loss}} = \lambda\, \|\boldsymbol{H}_\lambda \times \boldsymbol{\nu}\|^2_{\Gamma_{\rm loss}} .$$

Left-hand side of the last result can be rewritten into the more suitable form

$$(\boldsymbol{E} \times \boldsymbol{H}_\lambda, \boldsymbol{\nu})_{\Gamma_{\rm loss}} = \lambda\, \|\boldsymbol{H}_\lambda \times \boldsymbol{\nu}\|^2_{\Gamma_{\rm loss}} .$$

Using (1) one obtains $P = \lambda\, \|\boldsymbol{H}_\lambda \times \nu\|^2_{\Gamma_{\rm loss}}$, describing a physical relation between coefficient λ and iron loss P.

4. Estimates

First lemma guarantees the uniform estimate of $\boldsymbol{H}_\lambda$ in $\mathbf{H}(\mathrm{curl}\,; \Omega)$-norm and its trace with respect to $\lambda > 0$.

Lemma 4.1. *Let (8) be satisfied. Then, there exists a positive constant C such that*

$$\|\boldsymbol{H}_\lambda\|^2_{\mathbf{H}(\mathrm{curl}\,;\Omega)} + \lambda\, \|\boldsymbol{H}_\lambda \times \boldsymbol{\nu}\|^2_{\Gamma_{\rm loss}} \le C \quad \forall \lambda > 0.$$

Proof. The assertion can be readily proved by taking $\boldsymbol{\varphi} = \boldsymbol{H}_\lambda$ in (9) and using the Cauchy–Schwarz and Young's inequalities.

$$\begin{aligned} K\, \|\boldsymbol{H}_\lambda\|^2 + \|\nabla \times \boldsymbol{H}_\lambda\|^2 + \lambda\, \|\boldsymbol{H}_\lambda \times \boldsymbol{\nu}\|^2_{\Gamma_{\rm loss}} &\le C_\varepsilon\, \|\boldsymbol{f}\|^2 + \varepsilon\, \|\boldsymbol{H}_\lambda\|^2 \\ (K - \varepsilon)\, \|\boldsymbol{H}_\lambda\|^2 + \|\nabla \times \boldsymbol{H}_\lambda\|^2 + \lambda\, \|\boldsymbol{H}_\lambda \times \boldsymbol{\nu}\|^2_{\Gamma_{\rm loss}} &\le C_\varepsilon\, \|\boldsymbol{f}\|^2 \\ \|\boldsymbol{H}_\lambda\|^2_{\mathbf{H}(\mathrm{curl}\,;\Omega)} + \lambda\, \|\boldsymbol{H}_\lambda \times \boldsymbol{\nu}\|^2_{\Gamma_{\rm loss}} &\le C. \quad \square \end{aligned}$$

We introduce a real function $m(\lambda) : [0, \infty) \to [0, \infty)$ given by

$$m(\lambda) = \|\boldsymbol{H}_\lambda \times \boldsymbol{\nu}\|^2_{\Gamma_{\rm loss}} .$$

Hence, the function $m(\lambda)$ is defined in terms of the weak solution $\boldsymbol{H}_\lambda$ of (9).

Let us study first the behaviour of introduced function $m(\lambda)$.

Lemma 4.2. *Let (8) be satisfied. Then the function $m(\lambda)$ is continuous on* $(0, \infty)$.

Proof. Following the definition of continuity, $\lim_{\varepsilon\to 0}|m(\lambda)-m(\lambda+\varepsilon)|=0$ needs to be shown. Thus, let us fix any $\lambda>0$ and choose a small parameter ε satisfying $|\varepsilon|<\lambda$. Subtracting (9) from (9) for $\lambda=\lambda+\varepsilon$ one obtains

$$\begin{aligned} &K(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda,\boldsymbol{\varphi})+(\nabla\times(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda),\nabla\times\boldsymbol{\varphi})\\ &\quad+\lambda((\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\times\boldsymbol{\nu},\boldsymbol{\varphi}\times\boldsymbol{\nu})_{\Gamma_{\text{loss}}}+\varepsilon(\boldsymbol{H}_{\lambda+\varepsilon}\times\boldsymbol{\nu},\boldsymbol{\varphi}\times\boldsymbol{\nu})_{\Gamma_{\text{loss}}}=0. \end{aligned}\tag{10}$$

This can be rewritten to the equivalent form as

$$\begin{aligned} &K(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda,\boldsymbol{\varphi})+(\nabla\times(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda),\nabla\times\boldsymbol{\varphi})\\ &\quad+(\lambda+\varepsilon)((\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\times\boldsymbol{\nu},\boldsymbol{\varphi}\times\boldsymbol{\nu})_{\Gamma_{\text{loss}}}+\varepsilon(\boldsymbol{H}_{\lambda}\times\boldsymbol{\nu},\boldsymbol{\varphi}\times\boldsymbol{\nu})_{\Gamma_{\text{loss}}}=0. \end{aligned}\tag{11}$$

Summing up (10) and (11) and choosing $\boldsymbol{\varphi}=\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda$, the form

$$\begin{aligned} &2K\,\|\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda\|^2+2\,\|\nabla\times(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\|^2+(2\lambda+\varepsilon)\,\|(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\times\boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}\\ &\quad+\varepsilon((\boldsymbol{H}_{\lambda+\varepsilon}+\boldsymbol{H}_\lambda)\times\boldsymbol{\nu},(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\times\boldsymbol{\nu})_{\Gamma_{\text{loss}}}=0 \end{aligned}\tag{12}$$

is obtained. Using Lemma 4.1 for the last term on the left we deduce

$$\begin{aligned} \lim_{\varepsilon\to 0}\left|\varepsilon\left((\boldsymbol{H}_{\lambda+\varepsilon}+\boldsymbol{H}_\lambda)\times\boldsymbol{\nu},(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\times\boldsymbol{\nu}\right)_{\Gamma_{\text{loss}}}\right| &= \lim_{\varepsilon\to 0}|\varepsilon|\,\left|\|\boldsymbol{H}_{\lambda+\varepsilon}\times\boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}-\|\boldsymbol{H}_\lambda\times\boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}\right|\\ &\le \lim_{\varepsilon\to 0}C|\varepsilon|\left(\frac{1}{\lambda+\varepsilon}+\frac{1}{\lambda}\right)\\ &\le \lim_{\varepsilon\to 0}\frac{C|\varepsilon|}{\lambda}\\ &=0. \end{aligned}\tag{13}$$

Thus, the absolute value of the sum of the first three terms in (12) tends to 0 for $\varepsilon\to 0$. From the nonnegativity of each of these terms follows

$$\lim_{\varepsilon\to 0}\|\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda\|_{\mathbf{H}(\mathrm{curl}\,;\Omega)}=0\quad\text{and}\quad\lim_{\varepsilon\to 0}\|(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\times\boldsymbol{\nu}\|_{\Gamma_{\text{loss}}}=0.$$

Using Cauchy inequality, Lemma 4.1 and the last relation is finally obtained

$$\begin{aligned} \lim_{\varepsilon\to 0}|m(\lambda+\varepsilon)-m(\lambda)| &= \lim_{\varepsilon\to 0}\left|\|\boldsymbol{H}_{\lambda+\varepsilon}\times\boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}-\|\boldsymbol{H}_\lambda\times\boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}\right|\\ &= \lim_{\varepsilon\to 0}\left|\left((\boldsymbol{H}_{\lambda+\varepsilon}+\boldsymbol{H}_\lambda)\times\boldsymbol{\nu},(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\times\boldsymbol{\nu}\right)_{\Gamma_{\text{loss}}}\right|\\ &\le \lim_{\varepsilon\to 0}\|(\boldsymbol{H}_{\lambda+\varepsilon}+\boldsymbol{H}_\lambda)\times\boldsymbol{\nu}\|_{\Gamma_{\text{loss}}}\,\|(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\times\boldsymbol{\nu}\|_{\Gamma_{\text{loss}}}\\ &\le \frac{C}{\lambda}\lim_{\varepsilon\to 0}\|(\boldsymbol{H}_{\lambda+\varepsilon}-\boldsymbol{H}_\lambda)\times\boldsymbol{\nu}\|_{\Gamma_{\text{loss}}}\\ &=0, \end{aligned}$$

which proves the continuity of the function $m(\lambda)$. □

As a next step the monotonicity, more precisely decreasing behaviour of the function $m(\lambda)$ is to be proved.

Lemma 4.3. *Let (8) be satisfied. Moreover assume $\varepsilon>0$ and $\lambda>0$. Then, $m(\lambda+\varepsilon)\le m(\lambda)$.*

Proof. The first three terms in formula (12) are nonnegative and $\varepsilon>0$, thus, from the last term

$$m(\lambda+\varepsilon)=\|\boldsymbol{H}_{\lambda+\varepsilon}\times\boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}\le\|\boldsymbol{H}_\lambda\times\boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}=m(\lambda).$$

follows. □

Lemma 4.4 (*Asymptotic Character*). *Let (8) be satisfied. Then dollar $\lim_{\lambda\to\infty}m(\lambda)=0$.*

Proof. Using the result from Lemma 4.1

$$\lambda\,\|\boldsymbol{H}_\lambda\times\boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}\le C\quad\text{for }\forall\lambda>0,$$

is the assertion readily proved

$$\lim_{\lambda\to\infty}m(\lambda)=\lim_{\lambda\to\infty}\frac{\lambda\,\|\boldsymbol{H}_\lambda\times\boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}}{\lambda}=0.\quad\square$$

Theorem 4.1. *If the assumptions (8) are fulfilled and $\lambda>0$, then for any $0<M<\lim_{\lambda\to 0_+}m(\lambda)$ there exists a unique weak solution of the boundary value problem (9), (7).*

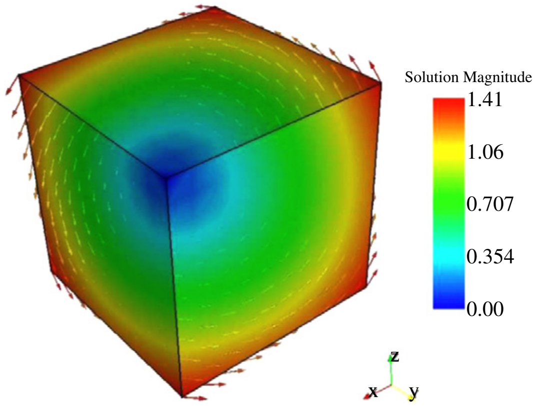

Fig. 1. Exact solution.

Proof. As a result from Lemmas 4.1–4.4 the existence of a weak solution has been guaranteed but we still have to show its uniqueness.

Suppose the existence of two solutions. Then one of the three following cases can occur:

(i*) Let $(\lambda, \boldsymbol{H})$ and $(\widetilde{\lambda}, \boldsymbol{H})$ be two different solutions of (9), (7). Subtracting the variational equations for both solutions from each other and setting the test function $\boldsymbol{\varphi} = \boldsymbol{H}$, one gets

$$(\lambda - \widetilde{\lambda})\ \|\boldsymbol{H} \times \boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}} = 0.$$

Hence, $\|\boldsymbol{H} \times \boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}} = 0$ contradicts with $M > 0$.

(ii*) Now, let $(\lambda, \boldsymbol{H})$ and $(\lambda, \widetilde{\boldsymbol{H}})$ be two solutions of (9), (7). Using the same steps as in previous case, but setting $\boldsymbol{\varphi} = \boldsymbol{H} - \widetilde{\boldsymbol{H}}$ one obtains

$$K \left\|\boldsymbol{H} - \widetilde{\boldsymbol{H}}\right\|^2 + \left\|\nabla \times (\boldsymbol{H} - \widetilde{\boldsymbol{H}})\right\|^2 + \lambda \left\|(\boldsymbol{H} - \widetilde{\boldsymbol{H}}) \times \boldsymbol{\nu}\right\|^2_{\Gamma_{\text{loss}}} = 0.$$

On account of (8) and of $\lambda > 0$ the last relation implies $\boldsymbol{H} = \widetilde{\boldsymbol{H}}$.

(iii*) Finally let $(\lambda, \boldsymbol{H}_\lambda)$ and $(\lambda + \varepsilon, \boldsymbol{H}_{\lambda+\varepsilon})$ with $\varepsilon > 0$ and $\lambda \geq 0$ solve (9), (7). Resulting from formula (12) and recalling that the each solution satisfies the side condition (7), i.e. $\|\boldsymbol{H}_{\lambda+\varepsilon} \times \boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}} = M = \|\boldsymbol{H}_\lambda \times \boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}}$ the equation yields to

$$2K \|\boldsymbol{H}_{\lambda+\varepsilon} - \boldsymbol{H}_\lambda\|^2 + 2 \|\nabla \times (\boldsymbol{H}_{\lambda+\varepsilon} - \boldsymbol{H}_\lambda)\|^2 + +(2\lambda + \varepsilon) \|(\boldsymbol{H}_{\lambda+\varepsilon} - \boldsymbol{H}_\lambda) \times \boldsymbol{\nu}\|^2_{\Gamma_{\text{loss}}} = 0. \tag{14}$$

This gives a contradiction, because the left-hand side is strictly positive due to $\boldsymbol{H}_\lambda \neq \boldsymbol{H}_{\lambda+\varepsilon}$.

The proof is done on the base of these three cases. □

5. Numerical experiment

Let Ω be a unit cube in $\mathbb{R}^3$. The boundary Γ is split up into two pieces as follows: on the bottom and the upper face of the cube is prescribed the iron loss boundary condition, on the other side faces the Neumann boundary condition is considered.

We apply our method to this test problem:

Find $(\lambda, \boldsymbol{H}_\lambda) \in (\mathbb{R}_+,\ \mathbf{H}(\text{curl}\,; \Omega))$ satisfying

$$\begin{aligned}
\boldsymbol{H}_\lambda + \nabla \times \nabla \times \boldsymbol{H}_\lambda &= \boldsymbol{f} && \text{in } \Omega \\
\nabla \times \boldsymbol{H}_\lambda \times \boldsymbol{\nu} &= \boldsymbol{g}_1 && \text{on } \Gamma_{\text{Neu}} \\
\nabla \times \boldsymbol{H}_\lambda \times \boldsymbol{\nu} &= \lambda(\boldsymbol{H}_\lambda \times \boldsymbol{\nu} \times \boldsymbol{\nu}) + \boldsymbol{g}_2 && \text{on } \Gamma_{\text{loss}} \\
\int_{\Gamma_{\text{loss}}} (\boldsymbol{H}_\lambda \times \boldsymbol{\nu})^2 &= 1.33,
\end{aligned}$$

where the data functions $\boldsymbol{f}$, $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ are defined such that

$$\lambda = 1.24$$

$$\mathbf{H}_\lambda = \begin{pmatrix} x_2 - x_1 \\ x_0 - x_2 \\ x_1 - x_0 \end{pmatrix}$$

is the exact solution (cf. Fig. 1).

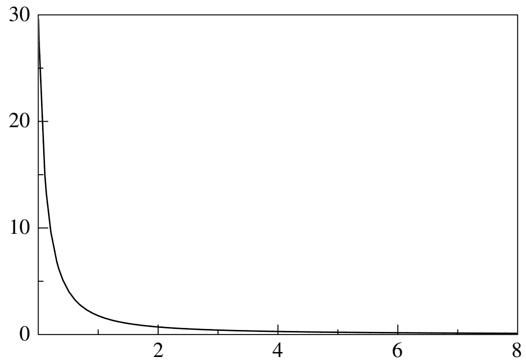

Fig. 2. The graph of the function $m(\lambda)$.

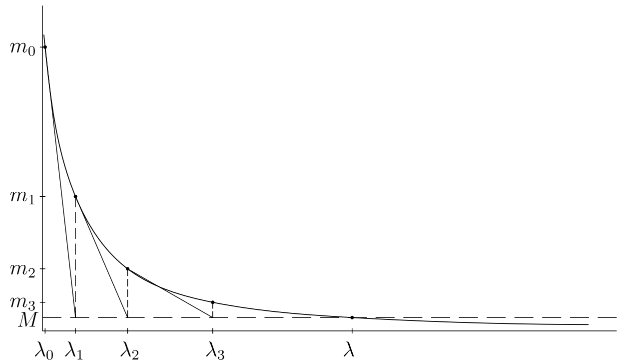

Fig. 3. Newton's method.

Fig. 2 shows the graph of the numerically obtained function $m(\lambda)$. For the determination of $\boldsymbol{H}_\lambda$ from the BVP (9) for each given λ Newton's method is used. Thus, we solve:

$$F(\boldsymbol{H}) = 0.$$

Starting with an initial guess $\boldsymbol{H}_0 = \boldsymbol{0}$ we compute

$$DF(\boldsymbol{H}_m)\boldsymbol{d}_m = F(\boldsymbol{H}_m) \quad \text{for } m > 0$$

and set

$$\boldsymbol{H}_{m+1} = \boldsymbol{H}_m - \boldsymbol{d}_m$$

until $\|\boldsymbol{d}_m\| < 1.0 \times 10^{-6}$. The functional $F(\boldsymbol{v})$ and its Fréchet derivative $DF(\boldsymbol{v})$ for $\boldsymbol{v} \in \mathbf{H}(\text{curl}\,; \Omega)$ are defined by

$$(F(\boldsymbol{v}), \varphi_j) = (\boldsymbol{v}, \varphi_j) + (\nabla \times \boldsymbol{v}, \nabla \times \varphi_j) + (\boldsymbol{g}_1, \varphi_j \times \boldsymbol{\nu})_{\Gamma_{\text{Neu}}} + \lambda(\boldsymbol{v} \times \boldsymbol{\nu}, \varphi_j \times \boldsymbol{\nu})_{\Gamma_{\text{loss}}} + (\boldsymbol{g}_2, \varphi_j \times \boldsymbol{\nu})_{\Gamma_{\text{loss}}} - (\boldsymbol{f}, \varphi_j)$$
$$(DF(\boldsymbol{v})\varphi_i, \varphi_j) = (\varphi_i, \varphi_j) + (\nabla \times \varphi_i, \nabla \times \varphi_j) + \lambda(\varphi_i \times \boldsymbol{\nu}, \varphi_j \times \boldsymbol{\nu})_{\Gamma_{\text{loss}}},$$

where $\varphi_i, \varphi_j \in \boldsymbol{V}$.

On our numerical scheme FEM is applied. The computational domain is split into 384 tetrahedrons with the mesh diameter $h = 0.433$. There is no need to split the domain into more subdomains. Due to the linearity of our problem this 'coarse mesh' is suitable enough to reach the satisfying accuracy of our model. For the approximation of the magnetic field $\boldsymbol{H}_\lambda$ Whitney's edge elements (cf. [4,5]) are used.

Newton's method is chosen again to determinate the Robin coefficient λ for which the iron loss boundary condition is satisfied. Here, the next approximation is given by:

$$\lambda_{\text{new}} = \lambda_{\text{old}} - \frac{m(\lambda)}{m'(\lambda)},$$

where

$$m'(\lambda) = \frac{m(\lambda + h) - m(\lambda - h)}{2h}$$

with $h = 0.005$.

Fig. 3 and Table 1 show the convergence of Newton's method. We have started with $\lambda = 0.01$ and the algorithm has stopped after five iterations with the prescribed precision $|m(\lambda) - 1.33| < 0.0001$. The following errors have been obtained

Table 1
Newton's iterations.

Iter.	λ	$m(\lambda)$	Error in %
1	0.010	28.379	127.47
2	0.132	13.438	48.68
3	0.341	6.206	14.40
4	0.682	2.858	2.40
5	1.240	1.330	0.00

for the last approximation:

$$\left\|\boldsymbol{H}_\lambda - \boldsymbol{H}_{\lambda\mathrm{app}}\right\|_{\mathbf{L}_2(\Omega)} = 5.920896 \times 10^{-09}$$
$$\left\|\boldsymbol{H}_\lambda - \boldsymbol{H}_{\lambda\mathrm{app}}\right\|_{\mathbf{H}(\mathrm{curl}\,;\Omega)} = 3.180161 \times 10^{-08}$$
$$|\lambda - \lambda_{\mathrm{app}}| = 4.29 \times 10^{-4}.$$

6. Conclusions

We have proved the well-posedness of the inverse recovery problem in static electromagnetism. The efficiency of the numerical method has been tested by numerical experiments.

The problem for determining an unknown information on an inaccessible part of the boundary has a large potential towards which our solution can be developed. In the future work we will consider λ not to be a coefficient but a function depended on space variable to achieve a problem with more applications in practice. Then the error analysis will be performed.

Our source of motivation is the work of Slodička and Van Keer which approaches the parabolic (cf. [6]) and the elliptic boundary value problem (cf. [7]) in this way.

Acknowledgement

The first author is supported by grant number 3G008206 of the Scientific Research Foundation - Flanders.

References

[1] M. Slodička, R. Van Keer, Determination of the convective transfer coefficient in elliptic problems from a nonstandard boundary condition, in: J. Maryška, M. Tůma, J. Šembera (Eds.), Simulation, Modelling, and Numerical Analysis, Technical University of Liberec, Liberec, 2000, pp. 13–20.
[2] G. Inglese, An inverse problem in corrosion detection, Inverse Problems 13 (4) (1997) 977–994.
[3] P. Monk, Finite element methods for Maxwell's equations, in: Numerical Mathematics and Scientific Computation, Oxford University Press, Oxford, 2003.
[4] M. Cessenat, Mathematical Methods in Electromagnetism, Linear Theory and Applications, in: Series on Advances in Mathematics for Applied Sciences, vol. 41, World Scientific Publishers, Singapore, 1996.
[5] A. Bossavit, Computational Electromagnetism, Variational Formulations, Complementarity, Edge Elements, in: Electromagnetism, vol. XVIII, Academic Press, Orlando, FL, 1998.
[6] M. Slodička, R. Van Keer, Determination of a robin coefficient in semilinear parabolic problems by means of boundary measurements, Inverse Problems 18 (1) (2002) 139–152.
[7] M. Slodička, R. Van Keer, A numerical approach for the determination of a missing boundary data in elliptic problems, Applied Mathematics and Computation 147 (2) (2004) 569–580.